Modelling the Trojan Asteroids

Christopher Gallagher University of Cambridge

April 26, 2020

Trojan asteroids orbit the Sun at Lagrange points in resonance with Jupiter, and undergo stable orbits about these points. This can be treated as a restricted three-body problem, which is numerically evaluated here using a stiff Radau method to solve the system of coupled differential equations. Without perturbations, a maximal deviation of 4.68×10−13 AU from the Lagrange points was observed over 1000 orbits (11,852 years). Applying perturbations to these asteroids demonstrated the predicted existence of stable tadpole and horseshoe orbits, and suggested wander from the Lagrange point depends quadratically on perturbation size. Such orbits were observed to be stable up to a planetary mass of 0.04 M, also in agreement with theoretical predictions. Word Count: XXXX

1 Introduction

The Jupiter trojans, commonly known as the Trojan asteroids, are two large groups of asteroids that share the planet Jupiter’s orbit around the Sun in a 1:1 orbit resonance. These two groups are called the Greeks and the Trojans, named after opposing sides in the mythological Trojan war, and lead/trail Jupiter respectively in its orbit. They correspond to Jupiter’s two stable Lagrange points: L4, lying 60° ahead of the planet in its orbit, and L5, 60° behind, with asteroids distributed in two elongated, curved regions around these Lagrangian points.

The first Jupiter trojan, 588 Achilles, was discovered in 1906 by the German astronomer Max Wolf [1], and a total of 7642 Jupiter trojans have been found as of February 2020 [2].

Research into Jupiter’s trojan asteroids continues, with the particular focus on their origins reliant on an understanding of their orbit stability [3], [4]. This informs studies into their composition [5], as travel to these asteroids is considered for their potential in mineral mining [6] [7].

The purpose of this report is to use numerical simulation techniques to investigate the stability of orbits about these Lagrange points, demonstrating the asteroid oscillate about these points under small perturbations and quantifying the absolute distance of the asteroids from the Lagrange point (the wander) during their orbits. The impact of variation in planetary/solar mass on asteroid orbit stability will also be considered.

2 Theoretical Background

2.1 Lagrange Points

The asteroids exist at/near Lagrange points, defined in Lagrange’s initial analysis of the threebody problem in 1772 [8], where he demonstrated the existence of five equilibrium points for an object of negligible mass orbiting under the gravitational effect of two larger masses. Three of these equilibrium points, L1-L3 lie on the line joining the two masses, while each of the remaining two points, L4 and L5, lie at the apex of an equilateral triangle with base equal to the separation of the two masses (see Figure 1). Despite all these points being potential maxima, stable motion is possible around L4 and L5 due to the Coriolis force [9].

Figure 1: The location of the five Lagrange equilibrium points in the circular-restricted three-body problem. The solar and planetary masses are denoted by the large and small filled circles, and the letters P, H, and T denote passing, horseshoe, and tadpole orbits respectively. Note that the two masses form an equilateral triangle with each of the L4 and L5 points. Reproduced from Marzari et al. [10]

Figure 1 depicts orbits about L4 and L5 (known as tadpole orbits), as well as orbits between Lagrange points (such as horseshoe orbits), described by Murray et al. [11].

This report will focus on tadpole orbits, a well documented feature of Trojan orbits [12], [13], and the restricted three-body problem in general. Their distinctive shape results from a long-period motion about the equilibrium point combining with a short-period oscillation due to Keplerian motion of the asteroid.

Szebehely et al. [14] predict this short period tends to the planetary period in the small planetary mass limit, while the long period is given by Tlong = TP r 4 27µ2 , (1) where µ2 = m2/(m1+m2) and TP is the period of planetary orbit.

2.2 Theoretical Model

The three-body problem, where the dynamics of three interacting bodies are determined given their initial positions and velocities, has no analytical (closed-form) solution in the general case [15].

In this report, I will consider the circular, restricted, three-body problem, where two of the bodies move in circular, coplanar orbits about their common centre of mass (CoM), unaffected by the negligible mass of the third body. I will also assume that all interactions are via Newtonian gravity.

The system of differential coordinates determine the position and velocity of the asteroids, with two equations per spatial coordinate. dri dt = vi , dvi dt = gi , i = x, y (2)

In this, gi is given by: g = − GMs |r − rs| 3 (r − rs) − GMp |r − rp| 3 (r − rp) (3) where the subscripts s and p refer to solar and planetary properties respectively.

We may also consider a frame rotating at the same speed as the massive bodies. As there is 1:1 orbital resonance between Jupiter and the asteroids, all three bodies are stationary in this frame. This significantly increases the accuracy of numerical simulations, as the exact solution is stationary with no explicit time dependence, rather then requiring an infinite power series [16], [17].

When transforming into this rotating noninertial frame, gi gains an additional virtual force term with coupling between the spatial coordinates. This is given below as the sum of the centripetal and Coriolis forces: ∆gi = Ω2 ri − 2[Ω × v]i (4) where Ω is the angular speed of the rotating frame, and v is the velocity of the asteroid within this frame.

2.3 Symmetry

This problem contains a number of symmetries, which were employed to simplify the problem and reduce the computational load. The Trojan and Greek asteroids are in equivalent positions, so experience the same forces, and the system is rotationally and inversion symmetric, so the choice of initial point and orbit direction is arbitrary. Therefore only the Greeks, orbiting counter-clockwise

Figure 2: A geometric depiction of the three-body system, in the case where the planet has a mass equal to a third of the sun, and asteroids are considered at the L4 point. Rs and Rp denote the (fixed) radii from the centre of mass (the grey point) to the Sun and the planet respectively, while ra denotes the radius of the asteroids.

with perturbations applied at t = 0, need be considered.

2.4 Orbit Geometry

As the three bodies considered here form an equilateral triangle in the initial equilibrium state, as depicted in Figure 2, we can derive the polar coordinates of each body with respect to the centre of mass about which the bodies orbit.

Using standard trigonometric relations, it is simple to show that the values ra and θ are given by: ra = q a 2 + RsRp, θ = tan−1 a sin( π 3 ) Rp − a 2 (5)

Furthermore, the Lagrange point in Cartesian coordinates based about the CoM is easily found to be: (x, y) = Rp − a 2 , √ 3a 2 ! (6)

Finally, equating the gravitational and centripetal forces on the planet allows the derivation of its (and all other bodies’) orbital velocity: Ω = r G(Ms + Mp) a 3 (7)

3 Methodology

This system of coupled first-order ordinary differential equations (ODEs) was solved using the scipy solve\_ivp function. The time span was taken as 100 orbits (with 100 points sampled per orbit) unless otherwise stated; this corresponds to 1185 Earth years. Rescaled solar system units are used for mathematical ease, so distances are measured in astronomical units (AU), time in earth years and mass in multiples of the solar mass, and to prevent floating point overflows due to the magnitude of the quantities considered in SI units.

The wander was defined as the maximum distance of the asteroid from the initial point during the orbit (for small perturbations this is also approximately the separation from the Lagrange point). Initial conditions are defined by the Lagrange point in each frame, with the initial velocity in the stationary frame defined by the period of Jupiter’s orbit, and split into Cartesian components.

3.1 Integration Method

Within solve\_ivp, the default solver is RK45 (an explicit Runge-Kutta method of order 5(4) [18]) and gives a deviation in asteroid position (from the Lagrange point) in the order of 10−4 AU in the rotating frame over 50 years. This is larger than expected, suggesting the system of equations may require an unreasonably small step size for numerical stability with respect to this method, even in regions where the solution curve is smooth [19]. Such systems are known as is stiff, and solvers designed for this typically do more work per step, allowing them to take much larger steps, and have improved numerical stability compared to nonstiff solvers such as RK45 [20].

Therefore the stiff "Radau" solver (an implicit Runge-Kutta method from the Radau IIA family of order 5 [21]) is used for increased stability [22], and achieves a deviation in asteroid position in the order of 10−13 AU instead. This also ensures stability in the stationary frame, with deviations of 0.76% in asteroid separation from Jupiter over 103 years, compared to 53% for the best non-stiff solvers. Page 3 of 10 Modelling the Trojan Asteroids

3.2 Programme Structure Global constants such as solar mass, and sun–planet separation, along with derived values from these such as orbital period and solar radius from the CoM, are given in an importable python module "constants".

Functions to evaluate these coupled differential equation systems are defined in module "orbits", while additional functions to evaluate the wander during the orbit (under different sampling routines) are implemented in "wander". Complete code listings of these files are given in Appendix C. Further files then import these modules and produce the plots given in this report, fully detailed in Appendix B.

To consider a varying planet mass, it was considered preferable to avoid reconstructing all functions to take this as an argument, as this requires re-evaluating all initial derived constants in the "orbits" module. Therefore, I iterate over alternative planetary masses, re-defining constant values in this instance, and directly import the required functions to compute the orbit.

3.3 Performance

Sampling 100 points per orbit for 100 orbital periods takes a mean time of 16.97 ± 34 ms, with sublinear scaling for sampling rate and approximately linear scaling with orbit number up to the array memory limit, achieved through the optimised integration routines within solve\_ivp. Computing the wander over position/velocity space with larger perturbations was, however, more time consuming, averaging 2.04 ± 0.16 s per point with the same parameters.

The main computational load was within the solve\_ivp function, which is already optimised well beyond the capabilities of the author. However, assumptions made in Section 2.3, such as only considering one set of asteroids, reduces this time somewhat, and vectorisation of other aspects of the orbit functions reduced the running time of the whole module. These approaches were not taken universally however, due to the size of the arrays required, and dominant effect on the ODE solver on the running time; instead, the orbit number was reduced when only a comparison of the maximum wander was required.

Figure 3: The wander of Greek asteroids from the Lagrange point in the stationary frame. Note the two oscillation components and constant maximum oscillation amplitude over time, demonstrating this point is indeed stable.

3.4 Fourier Analysis

For periodic oscillations/orbits, the Fast Fourier Transform (FFT) implemented in scipy is used to obtain values for period, with peak finding in scipy.signal identifying the exact values. The errors were estimated using the associated peak widths of each frequency component.

4 Results

4.1 Unperturbed Stability of Lagrange Points

Without perturbations applied, the Greeks’ orbit has a maximum deviation from L4 of 4.68 × 10−13 AU over 100 orbits (1185 years). This value is unchanged if 1000 orbits are considered instead, confirming the stability of this Lagrange point.

In the stationary frame, this wander from the (now moving) Lagrange point is depicted in Figure 3. The deviation oscillates with a magnitude of 9.10×10−2 AU, and a period of 148±4 years. This is modulated with a faster oscillation component of 11.85 ± 0.27 years, equivalent to the orbital period of the asteroids.

These much more significant errors are due to time dependence in the exact solution, as detailed in Section 2.2. Energy can also be evaluated, and conserved, in this inertial frame; asteroid specific energy varies within only 0.113% of the initial value, and with a similar periodicity to wander. Energy is also negative, confirming the asteroids are located within a bound orbit.

Animations produced to demonstrate the orbit in the stationary frame are included in Supplementary Material I-II. Animation I depicts the orbit as evaluated by the radau solver, while II depicts it with LSODA, demonstrating the drift present over time with non-stiff solvers.

4.2 Wander Analysis

Wander from the initial point was calculated for random perturbations with a maximum magnitude of 1% of the displacement from the origin, and the perturbation components parallel and perpendicular to the position vector from the CoM (hereafter referred to as radial and tangential components respectively) were considered separately. By considering these perturbations across position space, it was clear that the wander is fully determined by the radial component, with no tangential dependence (as shown in Figure 4).

Figure 4a demonstrates a polynomial dependence on perturbation size in the radial direction, with a negligible constant term. While a quadratic has been fitted here, it was not possible to eliminate the possibility of higher order terms, as increasing perturbation size beyond 0.06 AU can lead to unstable orbits.

We may consider the wander resulting from perturbations in position and velocity space. Figure 5a clearly shows the dependence on radial position perturbations only, while Figure 5b shows a similar the dependence on tangential velocity perturbations.

4.3 Orbit Types

Figure 6 displays orbits resulting from small perturbations in the radial and tangential directions.

The "tadpole" orbit in Figure 6a consists of two oscillating components, as described in Section 2.1. The short period, measured at 11.85 ± 0.61 years, is in excellent agreement with planetary orbital period as expected; meanwhile the long period is measured to be 148 ± 5 years, consistent with the analytical value of 144 years.

For a narrow range of larger perturbations, stable horseshoe orbits encompassing both Lagrange points may also be observed as depicted in Figure 7. This has a full period of 353 years, in agreement

Figure 4: Wander from an initial perturbation with maximum relative magnitude of 0.1% of the displacement from the origin, given in radial and tangential directions.

with Taylor et al.’s numerical result of 358±7 years [23].

4.4 Perturbations in Z direction

Perturbations in the Z direction are aligned with angular velocity vector for rotating frame, so experience no virtual forces in this direction, and oscillate under the influence of gravity alone. Considering small perturbations in the Z direction (so that the distance from the CoM can be considered constant), the period of such oscillations tend to the orbital period of the rotating frame under Newtonian gravity, detailed in Appendix A. These oscillations are observed to be approximately sinusoidal, with a measured period of 11.85 ± 0.48 years, in strong agreement with the theoretical predictions.

The overall wander, however, is observed to

Figure 5: Wander from an initial point in position and velocity space, only calculated over 50 orbits with 30 points per orbit to reduce computational load. Note the dependence on radial position and tangential velocity perturbation components.

follow Fig 8. It is suggested that, while the maximum deviation from the Lagrange point in the Z direction is simply twice the initial perturbation, wander in the XY plane increases quadratically. This can be initiated from a radial displacement due to the reduced gravitational force under the z perturbation, demonstrated in Fig 9, or more significantly from an initial perturbation in the XY plane. However, the oscillation of gravitational forces due to perturbations in the Z direction make this problem significantly more complicated and likely result in the observed deviations from the quadratic relations.

Figure 6: Orbits from an initial perturbation of magnitude 0.01 AU from the origin, in radial and tangential directions, over 12 orbital periods. Note the ’tadpole’ features resulting from the radial perturbation, and the significantly larger wander than from the tangential perturbation.

Figure 7: Horseshoe and tadpole orbits in the rotating frame, from radial perturbations of 0.07 and 0.01 AU respectively, over 30 orbital periods.

4.5 Variation of Planetary Mass The variation of wander with planetary mass is given in Figure 10. An approximate quadratic trend is initially observed in the small mass region, with minimum wander occurring at Mp = 0.026Ms. However the variation of wander below Mp = 0.04Ms in minimal, while orbits are unstable beyond this point, in agreement with theoretical predictions by Darwin [24].

5 Conclusion This report demonstrates the stability of Lagrange points L4 and L5, with a maximum deviation from the Lagrange points of 4.68 × 10−13 AU over 1000 orbits (11,852 years), in a frame rotating with the asteroids and without perturbations applied. Stability under perturbations in both the XY plane and in the Zdirection is also demonstrated, with

Figure 8: Variation of wander with increasing z perturbations. A strong linear component is observed initially in the un-perturbed case, however tends towards the perturbed quadratic case, while the linear offsets result from the initial perturbation in the XY plane.

Figure 9: Orbit of Greek asteroids following a 0.2 AU perturbation in the positive Z direction, over 12 orbital periods. Note the initial outwards radial displacement despite the lack of perturbation in the XY plane, and subsequent tadpole-like orbit.

Figure 10: Variation of maximum wander with planetary mass, under a 0.01 AU radial perturbation and 0.01 AU/year tangential boost. Note the orbit stability up to Mp = 0.04Ms, and instability beyond this.

a finite wander from these points independent of orbit duration. It was possible to further quantify this wander, suggesting a strong quadratic dependence on radial perturbation magnitude and independence of tangential perturbations (which did not grow beyond the initial perturbation size).

It was also possible to replicate both tadpole and horseshoe orbits from radial perturbations, and frequency components for these orbits in agreement with previous literature were obtained. It was also found that orbits become unstable when the planetary/- solar mass ratio is greater than 0.04, in agreement with theoretical predictions, and there is little mass dependence below this point.

Further work is required to evaluate coefficient values for polynomial relationships observed, and demonstrate a theoretical basis for such dependencies. Further research is also possible on the categorisation of different orbit types, and examination of initial conditions corresponding to formation mechanisms of these orbits.

Appendix A Z Perturbation Period

Figure 11 depicts a Z perturbation out of the XY plane, from the L4 point.

Figure 11: A geometric description of asteroids at the L4 point undergoing a perturbation of magnitude z0 out of the XY plane. The fixed equilibrium distance between the asteroids and the sun is denoted by a, while R represents the separation between the asteroids and the sun as a result of the perturbation

As the asteroid is equidistant from the sun and the planet, the magnitude of these forces depend on these bodies’ mass only, and gravitational force components from both massive bodies are similarly orientated in the Z direction. The total gravitational force acting on the asteroid is therefore given by:

F = − G(Ms + Mp)Ma R2 (8)

where Ms, Mp and Ma represent the solar, planetary and asteroid masses respectively. Resolving this force in Cartesian components, the Z component is given by:

Fz = F cos(φ) = F z R = − G(Ms + Mp)Maz R3 (9)

In conjunction with Newton’s Second Law, we therefore obtain a second order differential equation for z:

z¨ + G(Ms + Mp) R3 z = 0 (10)

which can be treated as simple harmonic motion in the small perturbation limit where R ≈ a. This has a angular frequency equal to (7):

Ω = r G(Ms + Mp) a 3 (11)

Appendix B Program Structure

Global constants, along with derived values from these are given in an python module "constants", from which relevant variable are imported in all other scripts.

Functions to evaluate these coupled differential equation systems are defined in module "orbits", while additional functions to evaluate the wander during the orbit are implemented in "wander". These modules are both imported into the scripts below for plotting and analysis, and given in the listings in Appendix C.

Other code files are listed by Section below in Table 1: